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Natural Frequencies of the Classical Two-Spin *XXZ* System

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An expression for the ratio of the natural frequencies of the classical two-spin *XXZ* system is determined. These are the frequencies of the angle variables and the expression does not require the system trajectories for its evaluation.

KEY WORDS: *XXZ* model; Classical two-spin system; natural frequencies; action–angle variables.

The classical two-spin model with uniaxially symmetric exchange coupling (*XXZ* model) is a two-dimensional integrable system for which analytical solutions for the system's trajectories have been determined.[(1)] The two natural frequencies of this system are those of the angle variables in an action–angle variable analysis. The ratio of these frequencies is significant in that its value as a rational or irrational number signifies periodic or quasiperiodic trajectories, respectively.

In this paper an expression for the ratio of the frequencies is given in a form which does not use the system trajectories as is true for the expressions for the frequencies reported previously.[(1)] ω_1 in the Appendix of ref. 1 is a time integral over a section of the trajectories.

In the following an expression for ω_1/ω_2 is obtained in terms of the coupling constants J and J_z and the constants of the motion E and M_z, using the notation of ref. 1. In the Appendix of that paper the Hamiltonian is given as

$$H = -J(S_1^x S_2^x + S_1^y S_2^y) - J_z S_1^z S_2^z$$

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and the following expression is given for ω_1:

$$\omega_1(M_z, E) = \tau_2^{-1} \int_0^{\tau_2} dt(\dot{\phi}_1 + \dot{\phi}_2)/2 \tag{1}$$

where $\tau_2 = 2\pi/\omega_2$ is the period of $\phi_1 - \phi_2$, S_1^z, and S_2^z. These variables have the same period since the expressions for E and M_z can be used to express any two variables in terms of the third, for example, $\phi_1 - \phi_2$ and S_2^z as functions of S_1^z. Furthermore, ϕ_1 and ϕ_2 have the same period since ϕ_1 and ϕ_2 each change by the same amount in a time τ_2 since that is the period of $\phi_1 - \phi_2$.

Therefore, since the integral in Eq. (1) is equal to the change in either ϕ_1 or ϕ_2 that occurs in one cycle of S_1^z, Eq. (1) may be rewritten as

$$\omega_1/\omega_2 = (2\pi)^{-1} \oint (d\phi_1/dS_1^z)\, dS_1^z \tag{2}$$

Rewriting the integrand in Eq. (2) as $\dot{\phi}_1/\dot{S}_1^z$ and using the equations of motion for these derivatives, the integrand can be obtained as a function of S_1^z. The result is

$$d\phi_1/dS_1^z = \pm [S_1^z(E - J_z S^2) + J_z M_z S^2]/[S^2 - (S_1^z)^2]\{J^2[S^2 - (S_1^z)^2] \times [S^2 - (M_z - S_1^z)^2] - [E + J_z S_1^z(M_z - S_1^z)]^2\}^{1/2} \tag{3}$$

where S is the magnitude of the spin vectors, and the + and − values are used when S_1^z is increasing and decreasing, respectively.

Integrating over a cycle of S_1^z in Eq. (2) can be changed to an integration from the minimum to the maximum value of S_1^z as follows:

$$\oint (d\phi_1/dS_1^z)\, dS_1^z = \int_{(S_1^z)^-}^{(S_1^z)^+} (d\phi_1/dS_1^z)^+ \, dS_1^z + \int_{(S_1^z)^+}^{(S_1^z)^-} (d\phi_1/dS_1^z)^- \, dS_1^z = 2\int_{(S_1^z)^-}^{(S_1^z)^+} (d\phi_1/dS_1^z)^+ \, dS_1^z \tag{4}$$

where the plus and minus signs on the parentheses in the integrands refer to the expression in Eq. (3) with the plus and minus signs, respectively. $(S_1^z)^+$ and $(S_1^z)^-$ are the maximum and minimum values of S_1^z, respectively. They can be obtained as functions of E and M_z by solving for $S_1^z(E, M_z, \cos^2(\phi_1 - \phi_2))$. The maximum and minimum values occur when $\dot{S}_1^z = 0$ or $\cos^2(\phi_1 - \phi_2) = 1$.

Finally, then,

$$\omega_1/\omega_2 = (\pi)^{-1} \int_{(S_1^z)^-}^{(S_1^z)^+} (d\phi_1/dS_1^z)^+ \, dS_1^z$$

where $(d\phi_1/dS_1^z)^+$ is the expression in Eq. (3) with the plus sign,

$$(S_1^z)^{\pm} = M_z/2 \pm (M_z^2 - 4F)^{1/2}/2 \tag{5}$$

and

$$F \equiv (J^2S^2 - J_zE \pm \{(J^2S^2 - J_zE)^2 - (J_z^2 - J^2) \times [E^2 - J^2S^2(S^2 - M_z^2)]\}^{1/2})/(J_z^2 - J^2)$$

REFERENCE

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